Cohomological properties of Hermitian symplectic threefolds

Grigory Papayanov

Abstract

A Hermitian symplectic manifold is a complex manifold endowed with a symplectic form ω, for which the bilinear form $\omega(I\cdot,\cdot)$ is positive definite. In this work we prove dd^c-lemma for 1- and (1,1)-forms for compact Hermitian symplectic manifolds of dimension 3. This shows that Albanese map for such manifolds is well-defined and allows one to prove Kählerness if the dimension of the Albanese image of a manifold is maximal.

Introduction

A Hermitian symplectic manifold is a complex manifold (M, I) together with a symplectic form ω, for which the bilinear form $\omega(I\cdot,\cdot)$ is positive definite (that is, $\omega(IX, X) > 0$ for any vector field X on M). Any Kähler manifold is obviously Hermitian symplectic, and it is an open problem whether there exist other examples of Hermitian symplectic manifolds. Hermitian symplectic manifolds were studied by Streets and Tian in [ST2] and [ST1]; they constructed an appropriate Ricci flow on Hermitian symplectic manifolds, and studied its convergency properties. Since then, many people searched for non-trivial examples of Hermitian symplectic manifolds.

The search for non-Kähler examples of Hermitian symplectic manifolds was vigorous, but ultimately unsuccessful. All common sources of examples of non-Kähler manifolds were tapped at some point.

For complex dimension 2, Hermitian symplectic structures are all Kähler. This was shown by Streets and Tian in [ST2]. Another proof could be obtained from the Lamari ([L]) result about existence of positive, exact $(1, 1)$-current on any non-Kähler complex surface.

In [Pe], it was shown that any non-Kähler Moishezon manifold admits an exact, positive $(n-1, n-1)$-current; therefore, Moishezon manifolds which are Hermitian symplectic are also Kähler.

In [EFV] it was shown that no complex nilmanifold can admit a Hermitian symplectic structure, and in [FKV] this result was extended to all complex solvmanifolds and Oeljeklaus-Toma manifolds.

Existence of Kähler metric implies some restrictions on the cohomology of a manifold: for example the Frölicher spectral sequence of Kähler manifold always degenerates at the first page. Results of Cavalcanti ([Ca]) show that the Frölicher spectral sequence for Hermitian symplectic manifolds degenerates at the first page.

In this work we define some Laplacian-like operators, kernels of which conjecturally isomorphic to the spaces of cohomology, and, with the help of these operators, prove dd^c-lemma for (1,1)-forms on Hermitian symplectic threefolds. Argument of Gauduchon ([G]) shows that dd^c-lemma for (1,1)-forms is equivalent to the equality $b^1 = 2h^{0,1}$. It follows that the Albanese map is well-defined and, if its image is not a point, the generic fiber of Alb is Kähler. The question of existence of special (e.g. Kähler or balanced) metrics on total spaces of maps with Kähler base and fibers is studied, for example, in [HL] and [Mi]. Using the Albanese map, we are able to prove that if a Hermitian symplectic threefold M has $\dim \mathrm{Alb}(M) = 3$, then it admits a Kähler metric, and if $\dim \mathrm{Alb}(M) = 1$, M is balanced. If dd^c-lemma holds for $(2,2)$-forms, then by [HL] $\dim \mathrm{Alb}(M) = 2$ would imply that M is Kähler, but, unfortunately, we have not proven dd^c-lemma in full generality yet.

Acknowledgements. The author would like to thank M. Verbitsky for many extremely helpful discussions. Work on sections 1–3 was supported by RSCF, grant number 14-21-00053, within the Laboratory of Algebraic Geometry. Work on section 4 was supported by RFBR 15-01-09242.

1 Preliminaries

Definition 1.1: Let M be a smooth manifold of dimension 2n, $I : TM \longrightarrow TM$ an integrable complex structure, $\mathcal{A}^{p,q}$ the corresponding Hodge decomposition on the bundle of differential forms: $\mathcal{A}^n \otimes \mathbb{C} = \bigoplus_{n=p+q} \mathcal{A}^{p,q}$, $\omega^{1,1}$ a form in $\mathcal{A}^{1,1}$. We will say that $\omega^{1,1}$ is *Hermitian* if the tensor $h(\cdot,\cdot) := \omega^{1,1}(\cdot, I\cdot)$ is a Riemannian metric on M, and we will say that $\omega^{1,1}$ is *Hermitian symplectic* if there exists a symplectic form ω such that $\omega^{1,1}$ is the (1,1)-component in the Hodge decomposition of ω. If M is endowed with such $\mathcal{I}$ and $\omega^{1,1}$, we will call it a Hermitian symplectic manifold.

For a Hermitian symplectic manifold (M, I, ω), let $d : \mathcal{A}^\bullet \longrightarrow \mathcal{A}^{\bullet+1}$ be the usual de Rham differential acting on forms, $d^c := IdI^{-1} : \mathcal{A}^\bullet \longrightarrow \mathcal{A}^{\bullet+1}$ the twisted differential, $L : A^\bullet \longrightarrow A^{\bullet+2}$ the operator of (left) multiplication by ω, $L(\eta) := \omega \wedge \eta$, $\Lambda : \mathcal{A}^\bullet \longrightarrow \mathcal{A}^{\bullet-2}$ the adjoint operator ([YT]). In the local Darboux coordinates p_i, q_i where $\omega = \sum dp_i \wedge dq_i$, operator Λ looks like $\sum i_{\frac{\partial}{\partial p_i}} i_{\frac{\partial}{\partial q_i}}$. We will denote by $L^{1,1}$ the operator of multiplication by the hermitian form $\omega^{1,1}$, and by $\Lambda^{1,1}$ the adjoint operator to $L^{1,1}$.

Lemma 1.2: The form $\omega^{1,1}$ is the SKT form, that is, $\partial\overline{\partial}\omega^{1,1} = 0$.

Proof: Let $\omega = \omega^{1,1} + \alpha$, where α lies in $\mathcal{A}^{2,0} \oplus \mathcal{A}^{0,2}$. Since $d\omega = 0$, $\partial\omega^{1,1} = -\overline{\partial}\alpha$ and $\partial\overline{\partial}\omega^{1,1} = \overline{\partial}^2\alpha = 0$. ■

Definition 1.3: Let α be a differential form on M. We will say that α is *primitive with*

respect to ω if $\Lambda\alpha = 0$, and that α is primitive with respect to $\omega^{1,1}$ if $\Lambda^{1,1}\alpha = 0$.

Lemma 1.4: (The Weil identities). Let $B^{p,q}$ be a primitive with respect to $\omega^{1,1}$ (p,q)-form, $p+q=r$. Then the following formula holds ([Vo, Proposition 6.29]):

$$*B^{p,q} = (-1)^{\frac{r(r+1)}{2}}(\sqrt{-1})^{p-q}\frac{1}{(n-r)!}(\omega^{1,1})^{n-k} \wedge B^{p,q}.$$

■

Definition 1.5: An operator Δ defined as double graded commutator, $\Delta := \{d, \{d^c, \Lambda^{1,1}\}\}$ is called *the Hermitian symplectic* Laplacian.

Remark 1.6: Δ is not a Laplacian associated to the Riemannian metric h. Nevertheless they differ by a differential operator of first order (see e.g. [LY] for the exact formula), therefore they have equal symbols, so Δ is elliptic.

Recall the graded Jacobi identity for the graded commutator:

$$\{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{deg(a)deg(b)}\{b, \{a, c\}\}.$$

Lemma 1.7: $\Delta = \{d^c, \{d, \Lambda^{1,1}\}\}$. Therefore Δ commutes with d and with d^c.

Proof: Follows simply from the Jacobi identity. ■

Theorem 1.8: (Spectral theorem). Let (M, I, ω) be a compact Hermitian symplectic manifold. Then the space of differential forms decomposes as a topological direct sum of generalized eigenspaces of Δ: $\mathcal{A}^\bullet(M) = \bigoplus_{\lambda_i} \mathcal{A}^\bullet_{\lambda_i}(M)$, each component of this decomposition is finite-dimensional and preserved by d, d^c and δ.

Proof: Decomposition is in fact proven in [BGV, Proposition 2.36] (Δ is a generalized laplacian in their terminology); one has to apply spectral theorem for compact operators: compact operator on Hilbert space has a canonical Jordan form with finite-dimensional generalized eigenvalues ([Co]).

By Lemma 1.7, Δ commutes with d and d^c, so all generalized eigenspaces are in fact subcomplexes. ■

Theorem 1.9: Let α be a closed form in $\bigoplus_{\lambda_i \neq 0} \mathcal{A}^\bullet_{\lambda_i}(M)$. Then α is exact.

Proof: When restricted to $\bigoplus_{\lambda_i \neq 0} \mathcal{A}^\bullet_{\lambda_i}(M)$, Laplacian Δ has an inverse, Δ^{-1}. So

$$\alpha = \Delta\Delta^{-1}\alpha = (\pm dd^c\Lambda \pm d\Lambda d^c)\Delta^{-1}\alpha.$$

■

2 Forms on a Hermitian symplectic manifold

In this section M is assumed to be compact.

Lemma 2.1: (dd^c-lemma for 1-forms). Let α be a d-exact, d^c-closed (or d^c-exact and d-closed) 1-form. Then $\alpha = 0$.

Proof: Suppose α is d-exact, $\alpha = df$. Then $dd^c f = 0$. By Hopf maximum principle ([GT]), f is constant, hence $\alpha = df = 0$. ■

We will now investigate whether holomorphic forms on M are closed.
Lemma 2.2: Let the n be the complex dimension of M. Then every holomorphic $n-2$-form is closed.

Proof: Let α be a holomorphic $n-2$-form, $\alpha \in \mathcal{A}^{n-2,0}$, $\overline{\partial}\alpha = 0$. Then $d\alpha = \partial\alpha$ is primitive with respec to $\omega^{1,1}$, by dimension reasons. So, by Weil identities,

$$||d\alpha||^2 = \int d\alpha \wedge d\overline{\alpha} \wedge \omega^{1,1} = \int \partial\alpha \wedge \overline{\partial}\overline{\alpha} \wedge \omega^{1,1} = \alpha \wedge \overline{\alpha} \wedge \partial\overline{\partial}\omega^{1,1} = 0.$$

Hence α is closed. ■

Remark 2.3: Obviously, on any compact complex manifold of complex dimension n, every holomorphic function and every holomorphic n-form is closed. Every holomorphic $n-1$-form is also closed, as the simple argument with the integration shows. So, any holomorphic form on a Hermitian symplectic threefold is closed.

3 dd^c-lemma for (1,1)-forms

Recall that by Theorem 1.8 every differential form α decomposes by generalized eigenspaces of Δ: $\alpha = \alpha_0 + \alpha_{\neq 0}$, where $\Delta^N(\alpha_0) = 0$ for some N, and $\alpha_{\neq 0} = \Delta\Delta^{-1}\alpha_{\neq 0}$. Suppose that α is d-exact and d^c-closed. Then α_0 and $\alpha_{\neq 0}$ are also d-exact and d^c-closed.

Lemma 3.1: In notations as above, $\alpha_{\neq 0}$ is dd^c-exact.

Proof: by Lemma 1.7, Δ^{-1} commutes with d and d^c, so $\Delta\Delta^{-1}\alpha_{\neq 0} = dd^c\Lambda^{1,1}\Delta^{-1}\alpha_{\neq 0} = \alpha$. ■

Lemma 3.2: Suppose exact (1,1)-form $\eta = d\gamma$ lies in the kernel of $\Delta^{1,1}$. Then η is primitive (with respect both to ω and to $\omega^{1,1}$).

Proof: $\Delta\eta = dd^c\Lambda^{1,1}\eta = 0$, so, by Hopf maximum principle [GT] $\Lambda^{1,1}\eta = c$, where c is some constant. It means that $\Lambda\eta$ also equals c.

If $\Lambda\eta = c$, then $\eta = c\omega + B$, where B is a primitive form. Since $\eta = d\gamma$, the cohomology classes of $c\omega$ and B are equal, but the cohomology class of a symplectic form cannot be represented by a primitive form. Indeed, $\omega \wedge \omega^{n-1}$ is a volume form, hence nonzero in cohomology, but $B \wedge \omega^{n-1} = 0$. So $c = 0$ and η is primitive. ■

Lemma 3.3: Suppose $\dim(M) = 3$, $\eta = dd^c f$ is dd^c-exact primitive $(1,1)$-form. Then $\eta = 0$.

Proof: Note first that, since η is primitive with respect to ω, it is primitive with respect to $\omega^{1,1}$, so, by Weil identities, $*\eta = \eta \wedge (\omega^{1,1})^{\wedge n-2}$, where $*$ is the Hodge star operator associated with the Hermitian metric h with corresponding 2-form equal to $\omega^{1,1}$ ([GH]). Then $h(\eta, \eta) =$

$$= \int \eta \wedge *\eta = \int \eta \wedge \eta \wedge (\omega^{1,1})^{\wedge n-2} = \int f\eta \wedge dd^c(\omega^{1,1})^{\wedge n-2}.$$

But $dd^c\omega^{1,1} = 0$ on a Hermitian symplectic manifold, so the integral vanishes. Since h is a hermitian metric, η also equals to zero. ■

Lemma 3.4: Let $\dim(M) = 3$. Suppose that an exact (1,1)-form $\eta = d\gamma$ lies in the kernel of $(\Delta)^n$, $n > 1$. Then η lies in the kernel of $(\Delta)^{n-1}$.

Proof: $(\Delta)^{n-1}\eta$ is an exact (1,1)-form lying in the kernel of Δ, so, by Lemma 3.2 it is primitive. Since $d\eta = d^c\eta = 0, (\Delta)^{n-1}\eta = (dd^c\Lambda)^{n-1}\eta$, it is dd^c-exact, therefore, by Lemma 3.3, it vanishes. ■

In order to complete the proof of dd^c-lemma for (1,1)-forms on Hermitian symplectic manifolds, we have to prove that an exact, primitive (1,1)-form vanishes.

Lemma 3.5: Let M be a Hermitian symplectic manifold of dimension 3, η be an exact, primitive (1,1)-form on M. Then $\eta = 0$.

Proof: Square of Hermitian norm of η is equal to $\int \eta \wedge \eta \wedge \omega^{1,1}$, but in dimension 3 we have the equality $\eta \wedge \eta \wedge \omega^{1,1} = \eta \wedge \eta \wedge \omega$; the latter form is exact, therefore $\eta = 0$. ■

Corollary 3.6: Let M be a compact Hermitian symplectic threefold, α is a d-closed, d^c-exact $(1,1)$-form. Then $\alpha = dd^c f$ for some function f. ■

4 Applications

Theorem 4.1: (Gauduchon, [G]). For a complex manifold M, dd^c-lemma for $(1,1)$-forms is equivalent to the equality $b^1 = 2h^{1,0}$.

Proof: Consider the cohomology sequence associated to the short exact sequence of sheaves of the form $0 \longrightarrow \sqrt{-1}\mathbb{R} \longrightarrow \mathcal{O} \xrightarrow{Re} \mathcal{H} \longrightarrow 0$, where $\mathcal{H}$ is the sheaf of pluriharmonic functions. The relevant piece looks like

$$... \xrightarrow{0} H^1(M, \sqrt{-1}\mathbb{R}) \longrightarrow H^1(M, \mathcal{O}) \longrightarrow H^1(M, \mathcal{H}) \longrightarrow H^2(M, \sqrt{-1}\mathbb{R}) \longrightarrow ...$$

It is well-known that

$$H^1(M, \mathcal{H}) = \frac{\operatorname{Ker} d : \mathcal{A}^{1,1} \longrightarrow \mathcal{A}^3}{\operatorname{Im} dd^c : \mathcal{A}^0 \longrightarrow \mathcal{A}^{1,1}}.$$

So dd^c-lemma for $(1,1)$-forms holds if and only if the third arrow is an isomorphism, and, by exactness, if and only if the first arrow is an isomorphism. ■

So, the equality $b^1 = 2h^{1,0}$ holds on compact Hermitian symplectic threefolds.

It follows that we have the Hodge decomposition on the first cohomology of M: $H^1(M, \mathbb{C}) = H^{0,1}(M) \oplus H^{1,0}(M)$, and $\dim H^{0,1} = \dim H^{1,0}$. So the rank of the abelian group $H^1(M, \mathbb{Z})$ is equal to the dimension of the real vector space $H^{0,1}(M)$. It follows that the Albanese torus is defined correctly and we have the Albanese map $\operatorname{Alb} : M \longrightarrow H^{0,1}(M)^*/H_1(M, \mathbb{Z})$. Its image $\operatorname{Alb}(M)$ is a subvariety (possibly singular) of a torus.

Proposition 4.2: Suppose $\dim \operatorname{Alb}(M) = 3$. Then M is Kähler.

Proof: If $\operatorname{Alb}(M)$ is smooth, then Alb is an immersion, and pullback of the Kähler form $\operatorname{Alb}^* \omega$ is the Kähler form on M. Otherwise, we can desingularize the morphism Alb to obtain the Kähler metric on some manifold $\tilde{M}$ bimeromorphic to M (M is then a manifold in the Fujiki class C). On the other hand, M admits an SKT structure (Lemma 1.2). From the theorem of Chiose ([Ch]) it follows that M is Kähler. ■

Remark 4.3: It would be interesting to know what one can extract from the Albanese map if $\dim \mathrm{Alb}(M) = 1$ or 2. For example, if $\mathrm{Alb}(M)$ is a smooth curve C, fibers of $\mathrm{Alb}(M)$ are Hermitian symplectic (and therefore Kähler) surfaces, and the pullback of the volume form $\mathrm{Alb}^* \mathrm{Vol}_C$ is a closed, non-exact $(1,1)$-form on M. By dd^c-lemma for $(1,1)$-forms and Remark 2.3, it could not be cohomologous to a form of type $(2,0)+(0,2)$. By a theorem of Michelsohn ([Mi]), in that situation there exists a *balanced* metric on M, that is, a Hermitian form ω such that $d\omega^{\dim M - 1} = 0$. Actually, the smoothness of C is not necessary, because a manifold bimeromorphic to a balanced manifold is balanced itself ([AB]).

References

[AB] L. Alessandrini, G. Bassanelli, *Modifcations of compact balanced manifolds*, In: C.R. Acad. Sci. Paris Math. **320** (1995), 1517–1522

[BGV] Nicole Berline, Ezra Getzler, and Michèle Vergne, *Heat Kernels and Dirac Operators*, Grundlehren Math. Wiss., vol. 298, Springer-Verlag, New York, 1992.

[Br] Brylinski, Jean-Luc; *A differential complex for Poisson manifolds*, In: J. Differential Geom. Volume 28, Number 1 (1988), 93–114.

[Ca] Gil R. Cavalcanti, *Hodge theory and deformations of SKT manifolds*, arXiv:1203.0493

[Ch] Chiose, I.; *Obstructions to the existence of Kähler structures on compact complex manifolds*, In: Proc. of the Amer. Math. Soc., **142** (2014), no. 10, 3561–3568.

[Co] John B. Conway, *A course in Functional Analisys*, Graduate Texts in Mathematics 96, Springer 1990.

[DGMS] Deligne, Pierre; Griffiths, Phillip; Morgan, John; Sullivan, Dennis; *Real homotopy theory of Kähler manifolds*, In: Invent. Math. **29** (1975), no. 3, 245–274.

[EFV] Enrietti, Nicola; Fino, Anna; Vezzoni, Luigi; *Tamed symplectic forms and SKT metrics,* arXiv:1002.3099.

[FKV] Fino, Anna; Kasuya, Hisashi; Vezzoni, Luigi; *Tamed complex structures on solvmanifolds*, arXiv:1302.5569.

[G] Gauduchon, Paul; *La 1-forme de torsion d'une variété hermitienne compacte*, Math. Ann. 267, 495-518 (1984).

[GT] D. Gilbarg,N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order,* Springer-Verlag, 1983

[GH] Griffiths, Phillip; Harris, Joe; *Principles of Algebraic Geometry*, Wiley-Interscience, New York, 1978.

[HL] Harvey, Reese; Lawson, H. Blaine; *An intrinsic characterization of Kähler manifolds*, Invent. math. **74**, 169–198 (1983).

[L] A. Lamari. *Courants kählériens et surfaces compactes* (French) [Kähler currents and compact surfaces]. Ann. Inst. Fourier (Grenoble) **49** (1999), no. 1, 263–285.

[LY] Kefeng Liu, Xiaokui Yang; *Geometry of Hermitian manifolds,* arXiv:1011.0207

[Mi] Michelsohn, M. L., *On the existence of special metrics in complex geometry.* Acta Math. 149 (1982), no. 3-4, 261–295.

[Pe] Peternell, Th., *Algebraicity criteria for compact complex manifolds.* Math. Ann. 275 (1986), no. 4, 653-672.

[ST1] Streets, Jeffrey; Tian, Gang; *Symplectic curvature flow*, arXiv:1012.2104.

[ST2] Streets, Jeffrey; Tian, Gang *A parabolic flow of pluriclosed metrics*, Int. Math. Res. Not. IMRN 2010, no. 16, 3101-3133.

[Vo] Voisim, Claire; *Hodge Theory and Complex Algebraic Geometry I: Volume 1*, Cambridge Studies in Advanced Mathematics.

[YT] Tseng, Li-Sheng; Yau, Shing-Tung; *Cohomology and Hodge Theory on Symplectic Manifolds: I*, arXiv:0909.5418.

G.P.: Laboratory of Algebraic Geometry,
National Research University HSE,
Department of Mathematics, 7 Vavilova Str. Moscow, Russia,
datel@mail.ru.